Towards a formally path-consistent Roe scheme for the six-equation, two-fluid model

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Abstract. We start from the most common formulation of the six-equation two-fluid model, from which we remove the non-conservative temporal term using a equivalent formulation derived in the literature. We derive a partially analytical, formally path-consistent Roe scheme, using the flux-splitting method.

We first expose the model in detail, and split the flux into a convective part, a pressure part, and a non-conservative part. Then, we derive an analytical Jacobian matrix of the fluxes, which allows the model to be written in quasilinear form. Finally, we explain the approach used to express formulas for the Roe-averaging of the variables. Only a simplified Roe-condition on the pressure remains. It can be fulfilled numerically, given any equation of state.

In the present article, we do not show the full results, but rather explain the approach. The full results will be explained at the conference.

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INTRODUCTION: THE MODEL

The six-equation two-fluid model [1, 3] is a well-studied two-phase flow model. In its most common formulation, without regularising term to force hyperbolicity, it takes the general form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \tilde{\mathbf{A}}(\mathbf{U})\frac{\partial \tilde{\mathbf{V}}(\mathbf{U})}{\partial t} + \tilde{\mathbf{B}}(\mathbf{U})\frac{\partial \tilde{\mathbf{W}}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}). \tag{1}$$

As described in [1], the non-conservative temporal term $\partial_t \tilde{\mathbf{V}}$ presents technical difficulties in deriving fully upwind schemes, as well as schemes that are *formally path-consistent* with respect to the definitions of the non-conservative products of the system.

In this work, we address this difficulty by taking advantage of a mathematically equivalent formulation, derived in [1], that eliminates the non-conservative temporal term. The system of equations is written

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \mathbf{B}'(\mathbf{U})\frac{\partial \mathbf{W}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}), \tag{2}$$

where the variables vector consists of the conserved quantities for each of the two phases (mass, momentum and total energy)

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} \rho_g \alpha_g \\ \rho_\ell \alpha_l \\ \rho_g \alpha_g v_g \\ \rho_\ell \alpha_l v_l \\ \rho_g \alpha_g \left(e_g + \frac{1}{2} v_g^2\right) \\ \rho_\ell \alpha_l \left(e_\ell + \frac{1}{2} v_l^2\right) \end{bmatrix}. \tag{3}$$

Further, the conservative flux $\mathbf{F}(\mathbf{U})$ is split into a convective part and a pressure part, such that

$$\mathbf{F} = \mathbf{F}_{\mathrm{c}} + \mathbf{F}_{\mathrm{p}} \quad \text{with} \quad \mathbf{F}_{\mathrm{c}}(\mathbf{U}) = \begin{bmatrix} \rho_{\mathrm{g}}\alpha_g v_g \\ \rho_\ell \alpha_l v_l \\ \rho_{\mathrm{g}}\alpha_g v_g^2 \\ \rho_\ell \alpha_l v_l^2 \\ \rho_{\mathrm{g}}\alpha_g v_g \left(e_{\mathrm{g}} + \frac{1}{2}v_g^2\right) \\ \rho_\ell \alpha_l v_l \left(e_\ell + \frac{1}{2}v_l^2\right) \end{bmatrix} \quad \text{and} \quad \mathbf{F}_p(\mathbf{U}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \alpha_g v_g p \\ \alpha_l v_l p \end{bmatrix}. \tag{4}$$

The term $\mathbf{B}'(\mathbf{U})\frac{\partial \mathbf{W}(\mathbf{U})}{\partial x}$ in (2) originally contains the non-conservative contributions of the fluxes, to allow using the formally path-consistent approach of Parés [5]. However, for simplicity of the analysis, $\mathbf{B}'(\mathbf{U})$ is modified to also contain the pressure part of the flux $\mathbf{F}_p(\mathbf{U})$, to give the system analysed in the present paper

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_c(\mathbf{U})}{\partial x} + \mathbf{B}(\mathbf{U})\frac{\partial \mathbf{W}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}), \tag{5}$$

where

$$\mathbf{B}(\mathbf{U}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \alpha_g & 0 \\ \alpha_l & 0 \\ \alpha_g v_g - \eta\,\alpha_g\alpha_l\,(v_g - v_l) & \eta\rho_\ell\alpha_g c_\ell^2 \\ \alpha_l v_l + \eta\,\alpha_g\alpha_l\,(v_g - v_l) & \eta\rho_{\mathrm{g}}\alpha_l c_{\mathrm{g}}^2 \end{bmatrix} \quad \text{and} \quad \mathbf{W}(\mathbf{U}) = \begin{bmatrix} p \\ \alpha_g v_g + \alpha_l v_l \end{bmatrix}, \tag{6}$$

using the abbreviation

$$\eta = \frac{p}{\rho_{\mathrm{g}}\alpha_l c_{\mathrm{g}}^2 + \rho_\ell\alpha_g c_\ell^2}. \tag{7}$$

Finally, the source term $\mathbf{S}(\mathbf{U})$ can represent gravity or phase interactions. Note that this formulation does not include regularising terms making the model hyperbolic, for which a number of possibilities exists in the literature. The numerical framework we present here may be extended to include such terms, following for instance the approach in [7].

QUASILINEAR FORM

In order to derive a Roe scheme [2], we first rearrange the model in the quasilinear form:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{S}(\mathbf{U}), \tag{8}$$

where

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}_c}{\partial \mathbf{U}} + \mathbf{B}(\mathbf{U})\frac{\partial \mathbf{W}}{\partial \mathbf{U}}. \tag{9}$$

Towards this aim, we first derive the analytical Jacobian matrix of the flux. A natural decomposition of the problem is to treat the convective part $\mathbf{F}_c$ separately from the rest of the flux, mainly involving the pressure, $\mathbf{B}(\mathbf{U})\frac{\partial \mathbf{W}(\mathbf{U})}{\partial x}$. The resulting Jacobian matrices will be called respectively $\mathbf{A}_c$ and $\mathbf{A}_p$.

The matrix $\mathbf{A}_c$ is

$$\mathbf{A}_c = \frac{\partial \mathbf{F}_c}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -v_g^2 & 0 & 2v_g & 0 & 0 & 0 \\ 0 & -v_l^2 & 0 & 2v_l & 0 & 0 \\ -v_g E_g & 0 & E_g & 0 & v_g & 0 \\ 0 & -v_l E_l & 0 & E_l & 0 & v_l \end{bmatrix} \quad \text{where} \quad E_\varphi = e_\varphi + \frac{1}{2}v_\varphi^2. \tag{10}$$

In order to derive the pressure part of the flux Jacobian $\mathbf{A}_p$, there is a need for the derivative of the non-conservative flux variables $\mathbf{W}$ with regard to the variable vector $\mathbf{U}$

(11)
$$\mathbf{M} = \frac{\partial \mathbf{W}(\mathbf{U})}{\partial \mathbf{U}} = \mathscr{R}^{-1} \begin{bmatrix} \zeta_l \beta_g & (v_g - v_l)\,\alpha_l \beta_g - \mathscr{R} v_g/\rho_g \\ \zeta_g \beta_l & (v_l - v_g)\,\alpha_g \beta_l - \mathscr{R} v_l/\rho_l \\ -\zeta_l \Gamma_g v_g & \mathscr{R}/\rho_g - (v_g - v_l)\,\alpha_l \Gamma_g v_g \\ -\zeta_g \Gamma_l v_l & \mathscr{R}/\rho_l - (v_l - v_g)\,\alpha_g \Gamma_l v_l \\ \zeta_l \Gamma_g & (v_g - v_l)\,\alpha_l \Gamma_g \\ \zeta_g \Gamma_l & (v_l - v_g)\,\alpha_g \Gamma_l \end{bmatrix}^T,$$

where

(12)
$$\begin{aligned} \beta_g &= c_g^2 - \Gamma_g \left(e_g + \tfrac{p}{\rho_g} \right) + \tfrac{1}{2}\Gamma_g v_g^2, \quad \zeta_g = \rho_g c_g^2 - \Gamma_g p, \quad \mathscr{R} = \alpha_g \zeta_l + \alpha_l \zeta_g, \\ \beta_l &= c_l^2 - \Gamma_l \left(e_l + \tfrac{p}{\rho_l} \right) + \tfrac{1}{2}\Gamma_l v_l^2, \quad \zeta_l = \rho_l c_l^2 - \Gamma_l p. \end{aligned}$$

We define

(13)
$$\mathbf{A}_p = \mathscr{R}\mathbf{B}(\mathbf{U}) \frac{\partial \mathbf{W}(\mathbf{U})}{\partial \mathbf{U}} = \mathscr{R}\mathbf{B}\mathbf{M}, \qquad \text{hence} \qquad \mathbf{A} = \mathbf{A}_c + \mathscr{R}^{-1}\mathbf{A}_p.$$

DERIVATION OF THE ROE SCHEME

The Roe scheme requires the construction of a matrix at each cell interface. It is the Jacobian matrix $\mathbf{A}$ evaluated at a particular average of the variables in the neighbooring cells. This will be called Roe averaging. It will be denoted in the following by $\hat{\mathbf{A}}$. It has to satisfy some conditions [2, 6, 7, 8], amongst which one is problematic:

R1: $\hat{\mathbf{A}}\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{U}^{\mathrm{R}} - \mathbf{U}^{\mathrm{L}}\right) = \mathbf{F}_c(\mathbf{U}^{\mathrm{R}}) - \mathbf{F}_c(\mathbf{U}^{\mathrm{L}}) + \overline{\mathbf{B}}\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{W}(\mathbf{U}^{\mathrm{R}}) - \mathbf{W}(\mathbf{U}^{\mathrm{L}})\right).$

The matrix $\overline{\mathbf{B}}$ is a property of the mathematical solution rather than the numerical method [1], and it is assumed that it is known in the present work.

Similarly to what was done in the derivation of the Jacobian matrix, we can split the problem into a convective part and a pressure part, such that

(14)
$$\hat{\mathbf{A}} = \hat{\mathbf{A}}_c + \hat{\mathscr{R}}^{-1}\hat{\mathbf{A}}_p.$$

The Roe condition R1 can subsequently be split in two, now reading

(15)
$$\hat{\mathbf{A}}_c\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{U}^{\mathrm{R}} - \mathbf{U}^{\mathrm{L}}\right) = \mathbf{F}_c(\mathbf{U}^{\mathrm{R}}) - \mathbf{F}_c(\mathbf{U}^{\mathrm{L}}),$$

(16)
$$\hat{\mathscr{R}}^{-1}\hat{\mathbf{A}}_p\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{U}^{\mathrm{R}} - \mathbf{U}^{\mathrm{L}}\right) = \overline{\mathbf{B}}\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{W}(\mathbf{U}^{\mathrm{R}}) - \mathbf{W}(\mathbf{U}^{\mathrm{L}})\right).$$

The derivation of the Roe matrix for the convective part $\hat{\mathbf{A}}_c$ is already well known, using the parameter vector approach of Roe [2]. Specifically, Toumi [3] gives the parameter vector for this case. On the other hand, this method is impractical for the pressure part. Instead, we follow a similar strategy as in [4]. It consists in reducing the partial Roe condition (16) on $\hat{\mathbf{A}}_p$ to two simpler ones. One will concern the pressure average, and the other the mixture velocity average. This opens for the possibility to construct a partially analytical Roe matrix for any equation of state.

From the averaging of (13) comes $\hat{\mathbf{A}}_p = \hat{\mathscr{R}}\overline{\mathbf{B}}\hat{\mathbf{M}}$. Here we remind that $\overline{\mathbf{B}}$ is known prior to the numerical method derivation, and does therefore not need Roe averaging. Insertion of $\hat{\mathbf{A}}_p$ into (16) gives

(17)
$$\hat{\mathbf{M}}\left(\mathbf{U}^{\mathrm{L}}, \mathbf{U}^{\mathrm{R}}\right)\left(\mathbf{U}^{\mathrm{R}} - \mathbf{U}^{\mathrm{L}}\right) = \mathbf{W}(\mathbf{U}^{\mathrm{R}}) - \mathbf{W}(\mathbf{U}^{\mathrm{L}}),$$

which results in a system of two equations. The matrix $\hat{\mathbf{M}}$ is the matrix $\mathbf{M}$ evaluated for specific weighted averages of the variables in the neighbouring cells, which we will call Roe-average and denote $\hat{x}$. For example, $\hat{v}_l$ is an average of v_l^{L} and v_l^{R}. We will use the system in question to define the Roe-averages of all the needed variables.

First, we show that the first line of (17) is fulfilled if we use a Roe-average of the pressure differential

(18)
$$\mathscr{R}\,\mathrm{d}p = \zeta_l \left(\frac{\zeta_g}{\rho_g} - \Gamma_g e_g \right) \mathrm{d}u_1 + \zeta_g \left(\frac{\zeta_l}{\rho_l} - \Gamma_l e_l \right) \mathrm{d}u_2 + \zeta_l \Gamma_g \,\mathrm{d}(u_1 e_g) + \zeta_g \Gamma_l \,\mathrm{d}(u_2 e_l),$$

as well as

$$u_5 = u_1 e_g + \frac{1}{2} u_1 v_g^2 \qquad \text{and} \qquad u_6 = u_2 e_l + \frac{1}{2} u_2 v_l^2, \tag{19}$$

and if we suppose that the velocities follow the usual Roe-averaging, weighted by $\left(\sqrt{\rho_\varphi \alpha_\varphi}\right)^{\mathrm{L,R}}$.

The condition expressed by the first line of (17) is then reduced to the condition found by Roe-averaging (18)

$$\begin{aligned}\hat{\mathscr{R}}(p^{\mathrm{R}} - p^{\mathrm{L}}) = \hat{\zeta}_l \left(\widehat{\frac{\zeta_g}{\rho_g}} - \hat{\Gamma}_g \hat{e}_g\right) \left((\rho_g \alpha_g)^{\mathrm{R}} - (\rho_g \alpha_g)^{\mathrm{L}}\right) + \hat{\zeta}_g \left(\widehat{\frac{\zeta_l}{\rho_l}} - \hat{\Gamma}_l \hat{e}_l\right) \left((\rho_l \alpha_l)^{\mathrm{R}} - (\rho_l \alpha_l)^{\mathrm{L}}\right) \\ + \hat{\zeta}_l \hat{\Gamma}_g \left((\rho_g \alpha_g e_g)^{\mathrm{R}} - (\rho_g \alpha_g e_g)^{\mathrm{L}}\right) + \hat{\zeta}_g \hat{\Gamma}_l \left((\rho_l \alpha_l e_l)^{\mathrm{R}} - (\rho_l \alpha_l e_l)^{\mathrm{L}}\right).\end{aligned} \tag{20}$$

Second, the same process is applied to the second line of (17). It is more involved, therefore we only show the results. We keep assumptions on the Roe-averaging of v_φ and we make further assumptions on the shape of the Roe-averages of α_φ and ρ_φ. Further, we define some other averaging formulas for $\breve{\alpha}$ and $\breve{\rho}$ which will be made explicit at the conference. Then we show that this second line will be reduced to the condition

$$\begin{aligned}\breve{\alpha}_g \hat{\alpha}_l \left(\widehat{\frac{\zeta_g}{\rho_g}}\right) \left((\rho_g)^{\mathrm{R}} - (\rho_g)^{\mathrm{L}}\right) - \breve{\alpha}_l \hat{\alpha}_g \left(\widehat{\frac{\zeta_\ell}{\rho_l}}\right) \left((\rho_l)^{\mathrm{R}} - (\rho_l)^{\mathrm{L}}\right) \\ + \breve{\rho}_g \breve{\alpha}_g \hat{\alpha}_l \hat{\Gamma}_g \left((e_g)^{\mathrm{R}} - (e_g)^{\mathrm{L}}\right) - \breve{\rho}_l \breve{\alpha}_l \hat{\alpha}_g \hat{\Gamma}_l \left((e_\ell)^{\mathrm{R}} - (e_\ell)^{\mathrm{L}}\right) = 0.\end{aligned} \tag{21}$$

Further, (20) and (21) reduce to the same condition by using the appropriate expression for $\widehat{\frac{\zeta_\varphi}{\rho_\varphi}}$

$$p^{\mathrm{R}} - p^{\mathrm{L}} = \frac{\hat{\zeta}_g}{\hat{\rho}_g} \left(\rho_g^{\mathrm{R}} - \rho_g^{\mathrm{L}}\right) + \breve{\rho}_g \hat{\Gamma}_g \left(e_g^{\mathrm{R}} - e_g^{\mathrm{L}}\right) = \frac{\hat{\zeta}_l}{\hat{\rho}_l} \left(\rho_l^{\mathrm{R}} - \rho_l^{\mathrm{L}}\right) + \breve{\rho}_l \hat{\Gamma}_l \left(e_l^{\mathrm{R}} - e_l^{\mathrm{L}}\right). \tag{22}$$

This last condition cannot be fulfilled analytically for a general equation of state. In case of non-analytical equation of state, or if its expression is too complicated, (22) will be the only condition that will be solved numerically. The approach presented in [4] can be used for example.

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